If $h_2 = h_1$ and $T_i = \frac{1}{2}(T_{i1} + T_{i2})$ this last equation becomes

$$V^2 = gh_1 \frac{T_{i2} - T_{i1}}{T_i}$$

If this approximate formula be applied to the preceding example as computed in §21, we have

$h_1 = 2000$ meters
$T_{i2} - T_{i1} = 3°$
$T_i = 261.62°$
} whence $V = 14.82$ meters per second.

CHAPTER III

METHODS OF COMPUTING THE AVAILABLE KINETIC ENERGY FOR A MASS OF AIR THAT PASSES ADIABATICALLY FROM ANY GIVEN INITIAL STAGE INTO THAT OF EQUILIBRIUM

§(23) Let there be given a mass of dry air, bounded by the horizontal ground plane and vertical walls, that is, initially at rest but not in equilibrium. We assume an initial condition of rest in order to be able to make use of the equation (α) of the preceding chapter, in our computation of the potential energy. We assume furthermore that that portion of the mass that is above the level surface h is already initially in equilibrium, and that during the changes that occur in the lower portion of the mass, this upper part acts like a piston of constant weight $p_h B$ where B is the area of the base or of any horizontal section.

Using a notation analogous to that of §13 we now have as before the following expression for the potential energy corresponding to the position of the column of air above the surface element dB

$$d\overline{P} = dB \left\{ \int_0^h p\,dz + (Z - h)\, p_h \right\}$$

Whence follows the $\overline{P}$ for the whole mass from the ground up to the level surface at the altitude h

$$\overline{P} = \int B \int_0^h p\,d\,B\,dz + B\,(Z - h)\,p_h = \int p\,dk + \text{Constant}$$
$$= R \int T\,dm + \text{Constant}.$$

Similarly for the initial stage we obtain

$$(\overline{P} + \overline{I})_a = C_p \int T\,dm + \text{Constant}$$

The constant on the right hand is the same in both the initial and final stages.

In the case of an adiabatic passage from the initial over to the final stage the temperature T of the elementary mass dm becomes T' and the total potential energy of the whole mass becomes

$$(\overline{P} + \overline{I})_e = C_p \int T' dm + \text{Constant}$$

Hence the available kinetic energy in the initial stage is

$$-\delta\,(\overline{P} + \overline{I}) = C_p \int (T - T')\, dm\,.$$

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§(24) *First analysis. The initial stage.* The mass of air in the chamber 1 of our fig. 1 is in neutral equilibrium, and S_1 is the entropy of the unit of mass. Similarly in the chamber 2 the air has the same volume but a higher entropy, S_2, and is in neutral equilibrium. The problem is wholly analogous to that treated previously, only the mass having the smaller entropy now lies alongside the other and not above it.

The given data of the present problem are: The temperatures T_{h1} and T_{h2} at the altitude h and the values B, h, p_h. From these we find the following initial temperature and pressures at the bases of the two masses.

$$T_{01} = T_{h1}\left(1 + \frac{gh}{C_p T_{h1}}\right)$$

$$T_{02} = T_{h2}\left(1 + \frac{gh}{C_p T_{h2}}\right)$$

$$p_{01} = p_h\left(\frac{T_{01}}{T_{h1}}\right)^{1/\kappa} = p_h\left(1 + \frac{gh}{C_p T_{h1}}\right)^{1/\kappa}$$

$$p_{02} = p\left(\frac{T_{02}}{T_{h2}}\right)^{1/\kappa} = p_h\left(1 + \frac{gh}{C_p T_{h2}}\right)^{1/\kappa}$$

From these and equation (5*a*) of the preceding chapter, section 20, there results

$$(\overline{P} + \overline{I})_a = C_p \cdot \frac{1}{g} \cdot \frac{1}{1+\kappa} \cdot \frac{B}{2} \cdot \{T_{01}\, p_{01} - T_{h1}\, p_h + T_{02}\, p_{02} - T_{h2}\, p_h\} + \text{Constant.}$$

The final stage. The mass 2′ of higher entropy now occupies the upper part of the whole volume of the trough, the mass 1 having

lower entropy is separted from it by the level suface i; at this level the temperature changes suddenly from T'_{i1} to T'_{i2}. Each of the two masses is individually in neutral equilibrium. At the upper surface of 2′ the pressure is p_h, consequently the temperature is T_{h2}. Since the entropies S_1 and S_2 remain unchanged therefore the values of the pressures at i and at the base (p'_2 and p'_0) are to be computed from the weight of the mass and thus we completely know the final stage, as follows:

Location.	Pressure.	Temperature.
At the upper boundary 2′	p_h	T_{h2}
At the surface of separation i between the masses 2′ and 1′	$p'_i = p_h + \frac{1}{2}(p_{02} - p_h)$	$T'_{i2} = T_{h2}\left(\frac{p'_i}{p_h}\right)^{\kappa}$ $T'_{i1} = T_{h1}\left(\frac{p'_i}{p_h}\right)^{\kappa}$
At the base level we have.....	$p'_0 = p_h + \frac{1}{2}(p_{02} - p_h) + \frac{1}{2}(p_{01} - p_h)$	$T'_0 = T_{h1}\left(\frac{p'_0}{p_h}\right)^{\kappa}$

We thus obtain all the quantities that enter into the expression for the total energy of the final stage

$$(\overline{\mathrm{P}} + \overline{\mathrm{I}})_e = C_p \cdot \frac{1}{g} \cdot \frac{1}{1+\kappa} \cdot B \left\{ T'_0 p'_0 - T'_{i1} p'_i + T'_{i2} p'_i - \right.$$
$$\left. - T_{h2} p_h \right\} + \text{Constant}$$

except only the arbitrary constant which will itself disappear when the difference is taken.

We assume that the available kinetic energy, viz.,

$$-\delta(\overline{\mathrm{P}} + \overline{\mathrm{I}}) = (\overline{\mathrm{P}} + \overline{\mathrm{I}})_a - (\overline{\mathrm{P}} + \overline{\mathrm{I}})_e,$$

belongs specifically to the mass M below the piston, and that this therefore may be written $\frac{1}{2} MV^2$.

In frictionless motion if the final stage be attained simultaneously by all the masses, then $\frac{1}{2}V^2$ is their average kinetic energy. Since $M = B\left(\frac{p'_0 - p_h}{g}\right)$ therefore V is independent of the area of the base.

Finally we compute the heights of the strata 1′ and 2′ from the temperatures by the formulæ

$$h'_1 = \frac{C_p}{g}(T'_0 - T'_{i1}) \qquad h'_2 = \frac{C_p}{g}(T'_{i2} - T_{h2})$$

Example 1. Given $h = 3000$ meters; $p_h = 510^{mm}$ mercury

$$T_{h1} = 243° \qquad T_{h2} = 248°$$

These data and the quantities computed from them for the initial and final stages are arranged in the following tabular form:

Initial stage

$h = 3000$ meters

$T_{h1} = 243°$ $\quad p_h = 510^{mm}$ $\quad T_{h2} = 248°$

$T_{01} = 272.8027°$ $\quad p_{01} = 759.1980$ $\quad p_{02} = 753.4621$ $\quad T_{02} = 277.8027°$

$$(\overline{P} + \overline{I})_a = \frac{C_p}{g} \cdot \frac{1}{1+\kappa} B \cdot p_h \times 162.7598 + \text{Constant}$$

Final stage

$$p_h = 510^{mm} \ldots \left\{ \begin{matrix} T_{h2} = 248° \\ T'_{i2} = 263.9266 \end{matrix} \right\} h'_2 = 1603.2 \text{ meters.}$$

$$p'_i = 631.7310$$

$$p'_0 = 756.3300 \quad \left\{ \begin{matrix} T'_{i1} = 258.6055 \\ T'_{01} = 272.5026 \end{matrix} \right\} h'_1 = 1398.9 \text{ meters.}$$

$$(\overline{P} + \overline{I}) = \frac{C_p}{g} \cdot \frac{1}{1+\kappa} \cdot B \cdot p_h \times 162.7127 + \text{Constant}$$

$$\tfrac{1}{2} V^2 = C_p \frac{1.41}{1.82} \times \frac{0.0471}{0.4830}$$

$$V = 12.2 \textit{ meters per second.}$$

Example 2 $\quad h = 3000^m \quad p_h = 510^{mm}$

$T_{h1} = 243°$ $\quad T_{h2} = 253°$ $\quad$ whence $V = 17.3$ *m/sec.*

Example 3 $\quad h = 6000^m \quad p_h = 325^{mm}$

$T_{h1} = 213°$ $\quad T_{h2} = 218°$ $\quad$ whence $V = 18.3$ *m/sec.*

Example 4 $\quad h = 6000^m \quad p_h = 325^{mm}$

$T_{h1} = 213°$ $\quad T_{h2} = 223°$ $\quad$ whence $V = 25.8$ *m/sec.*

§(25) *Second analysis:* approximate method for the case when the masses 1 and 2 are each initially in stable equilibrium and the entropy of the highest layer of 1 is smaller than that of the lowest layer of 2.

Since it is again assumed that every particle of the mass behaves adiabatically, therefore the layers in 1 retain their relative positions when they become 1′; similarly for the layers of 2 when they become 2′. We will designate the areas of the floors of the chambers by B_1 and B_2 so that $B_1 + B_2 = B$.

If p'_1 is the pressure in the final stage of any layer of 1′ which had the pressure p_1 when it was in the initial stage, then we have the relation

$$p'_1 = p_h + \frac{B_1}{B}(p_1 - p_h) + \frac{B_2}{B}(p_{02} - p_h) = p_1 + \frac{B_2}{B}(p_{02} - p_1)$$

Similarly when p'_2 and p_2 refer to another equal mass in the chambers 2′ and 2 in fig. 1 we have

$$p'_2 = p_h + \frac{B_2}{B}(p_2 - p_h) = p_2 - \frac{B_1}{B}(p_2 - p_h).$$

Hence the temperatures T'_1 and T'_2 of these masses in the final stage are to be computed from their initial temperatures T_1 and T_2 by the following equations:

$$T'_1 = T_1\left(\frac{p'_1}{p_1}\right)^\kappa = T_1\left(1 + \frac{B_2}{B}\cdot\frac{p_{02} - p_1}{p_1}\right)^\kappa$$

$$= T_1\left(1 + \kappa\frac{B_2}{B}\cdot\frac{p_{02} - p_1}{p_1}\right) \text{approximately.}$$

$$T'_2 + T_2\left(\frac{p'_2}{p_2}\right)^\kappa = T_2\left(1 - \frac{B_1}{B}\cdot\frac{p_2 - p_h}{p_2}\right)^\kappa$$

$$= T_2\left(1 - \kappa\frac{B_1}{B}\cdot\frac{p_2 - p_h}{p_2}\right) \text{approximately.}$$

The approximate formulæ hold good for every p_1 and p_2 when $p_{01} - p_h$ is small relative to p_h.

In the computation of the integrals that occur in the expression for the available kinetic energy we take layers of 1 and 2 as the

elementary masses and therefore by making use of the equation (α) § 13 we have

$$dm_1 = -B_1 \frac{dp_1}{g} \qquad dm_2 = -B_2 \frac{dp_2}{g}$$

Substituting hereafter only approximate values we obtain

$$\int (T_1 - T_1')\, dm_1 = -\frac{\kappa}{g} \cdot \frac{B_1 B_2}{B} \int_{p_h}^{p_{01}} T_1 \frac{p_{02} - p_1}{p_1} dp_1$$

$$\int (T_2 - T_2')\, dm_2 = +\frac{\kappa}{g} \cdot \frac{B_1 B_2}{B} \int_{p_h}^{p_{02}} T_2 \frac{p_2 - p_h}{p_1} dp_2$$

We may remark that although in the development of the binomials we previously retained only the terms of the first degree in $\frac{p_0 - p_h}{p_h}$ yet on account of the limits of these definite integrals, they are accurate to terms of the second degree.

From equation (α) there results

$$\int_{p_h}^{p_{01}} \frac{T_1}{p_1} dp_1 = \int_{p_h}^{p_{02}} \frac{T_2}{p_2} dp_2 = \int_0^h \frac{g}{R} dz = \frac{gh}{R}$$

If we introduce the average temperatures T_1^* and T_2^* of the masses 1 and 2 as determined by the equations

$$T_1^* (p_{01} - p_h) = \int_{p_h}^{p_{01}} T_1 dp_1; \qquad T_2^* (p_{02} - p_h) = \int_{p_h}^{p_{02}} T_2 dp_2$$

we thus obtain for the above integrals

$$\int (T_1 - T_1')\, dm_1 = \frac{\kappa}{g} \cdot \frac{B_1 B_2}{B} \left\{ T_1^* (p_{01} - p_h) - \frac{gh}{R} p_{02} \right\}$$

$$\int (T_2 - T_2')\, dm_2 = \frac{\kappa}{g} \cdot \frac{B_1 B_2}{B} \left\{ T_2^* (p_{02} - p_h) - \frac{gh}{R} p_h \right\}$$

These expressions may be still further simplified if we introduce other average temperatures $\odot_1$ and $\odot_2$ which for distinction we will

call barometric temperatures as determined by the barohypsometric relations*

$$p_{01} = p_h e^{\frac{gh}{R\odot_1}} = p_h\left(1 + \frac{gh}{R\odot_1} + \frac{1}{2}\left(\frac{gh}{R\odot_1}\right)^2\right) \text{ approximately.}$$

$$p_{02} = p_h e^{\frac{gh}{R\odot_2}} = p_h\left(1 + \frac{gh}{R\odot_2} + \frac{1}{2}\left(\frac{gh}{R\odot_2}\right)^2\right) \text{ approximately.}$$

In the serial development of these exponentials we have stopped at the terms of second degree in accordance with a preceding remark. With these values our integrals become†

$$\int (T_1 - T_1')\, dm_1 = \frac{\kappa}{g} \cdot \frac{B_1 B_2}{B} \cdot p_h \cdot \frac{gh}{R} \times$$

$$\times \left\{\frac{T_1^*}{\odot_1} - 1 - \frac{gh}{R}\left(\frac{1}{\odot_2} - \frac{T_1^*}{2\odot_1^2}\right)\right\}$$

$$\int (T_2 - T_2')\, dm_2 = \frac{\kappa}{g} \cdot \frac{B_1 B_2}{B} \cdot p_h \cdot \frac{gh}{R} \times$$

$$\times \left\{\frac{T_2^*}{\odot_2} - 1 + \frac{gh}{R}\frac{T_2^*}{2\odot_2^2}\right\}$$

Now in the actual applications to our atmosphere the quantities $T^* - \odot$ are always very small relative to $\frac{gh}{R}$. For the value $h = 3000$

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*The e is the base of the napierian logarithms.

†T = temperature of air at any time or place.

T^* = true average temperature of a mass of air as determined by the mass-relation,

$$T^* = \frac{\int_{p_h}^{p_o} T\, dp}{p_o - p_h}$$

$\odot$ = Approximate average temperature of mass of air as determined by the barohypsometric relation

$$p_o = p_h\, e^{gh/R\odot}$$

whence

$$\log \frac{p_o}{p_h} = \text{Modulus} \times \frac{gh}{R\odot}$$

$$\odot = \text{Mod.} \frac{gh}{R} \frac{1}{\log \frac{p_o}{p_h}}$$

meters this quantity is 102° and the difference between the true average temperature T^* and the barometric average temperature $\odot$ can only attain 2° Centigrade in extreme cases. For smaller values of h the $T^* - \odot$ is also smaller. We can therefore here substitute T^* for the corresponding $\odot$ and obtain

$$C_p \int (T_1 - T_1')\, dm_1 = -\frac{R}{g} \cdot \frac{B_1 B_2}{B} \cdot p_h \left(\frac{gh}{R}\right)^2 \left(\frac{1}{T_2^*} - \frac{1}{2T_1^*}\right)$$

$$C_p \int (T_2 - T_2')\, dm_2 = -\frac{R}{g} \frac{B_1 B_2}{B} p_h \left(\frac{gh}{R}\right)^2 \left(\frac{1}{2T_2^*}\right)$$

Finally by writing

$$(T^*)^2 = T_1^* T_2^*,$$

$$\frac{T_2^* - T_1^*}{T^*} = \tau,$$

$$Bp_h \frac{gh}{RT^*} = gM \text{ (approximately)}$$

we obtain the following expression for the available kinetic energy,

$$C_p \int (T - T')\, dm = \frac{MV^2}{2} = \frac{M}{2} \cdot \frac{B_1 B_2}{B^2}\, gh\tau.$$

V is independent of the constants R and C_p that characterize the physical properties of the gas. For a given value of B this expression for the available kinetic energy is greatest when $B_1 = B_2 = \frac{1}{2}B$; wherefore when the chambers 1 and 2 have equal volumes then the velocity is

$$V = \frac{1}{2}\sqrt{gh\tau} \quad \ldots\ldots\ldots\ldots \quad (I)$$

This approximate method suffices completely for the cases above given as examples of our first method of computation (see section 24). Since in the limiting case neutral equilibrium becomes stable equilibrium we may also apply this method of computation to those examples also. In the computation of T we may substitute $\frac{1}{2}(T_0 + T_h)$ for the average temperature. The values of V computed by the approximate formula for those four examples now become respectively

Example	*Velocity m. p. s.*
(1)	12.3
(2)	17.4
(3)	18.6
(4)	26.3

The assumptions of the approximate computation do not hold good for $h = 6000$ meters; but still the final formula gives very good approximate values; this is also true for systems having still greater altitudes.

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§(26) If the overturning of the masses 1 and 2 takes place under constant volume the computation proceeds in a perfectly similar manner. These masses considered by themselves now form the closed system to which equation (6**) of section 20 is to be applied. The diminution of $\overline{\mathrm{P}}$ will be greater than before and by so much smaller will be the diminution of $\overline{\mathrm{I}}$, so that we obtain the same value of V as for the overturning under constant pressure.

We may also state the problem thus: During the overturning the pressure at the upper boundary surface changes but has the same value throughout all parts of this surface. If the change proceeds so slowly that no appreciable amount of kinetic energy is thereby produced, then we again obtain the same equation (I) as just now deduced for the velocity V. This is not a case of a closed system. In place of the equation (6**) we now have

$$\delta\,\overline{\mathrm{K}} + (\overline{\mathrm{R}}) = -\,\delta\,(\overline{\mathrm{P}} + \overline{\mathrm{I}}) - B\int' p_h\,dh$$

where the first term on the right hand refers only to the masses 1 and 2 and can be written in the form

$$C_p\int(T - T')\,d\,m + B\,(h'\,p'_h - h\,p_h)$$

whereas the second term on the right results from the motion of the movable piston. When p_h is constant the sum of these right-hand terms becomes $C_p\int(T - T')dm$. The case of constant volume corresponds to $h' = h$.

We will now leave the two chambers system and compute the available kinetic energy in a special case of continuous distribution of temperature.

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§(27) *Third analysis.* The initial conditions are a continuous horizontal distribution of temperature and a vertical diminution of temperature corresponding to that of neutral equilibrium.

We assume that the trough is a parallelopipedon of air having a unit breadth and a length l along the horizontal axis of x, and that

in the initial stage the temperature of the lowest surface is a continuous function of the distance x from the left-hand end of the trough; therefore the temperature at the point (x, z) is expressed by the equation

$$T = f(x) - \frac{gz}{C_p}$$

If the $f(x)$ increases from the left toward the right then the entropy will also do so, since the pressure p_h is again assumed to be constant at the level h; hence the entropy is only a function of the distance x or the length of the trough and this is equally true of the pressure p_{0x} at the base level.

If the mass of air under the pressure p_h passes over adiabatically into the condition of equilibrium, then a horizontal layer is formed from each vertical column of the initial condition and the masses will now succeed each other from below upward in the same order as they were before arranged from the left toward the right. A mass originally in a vertical column at x having an initial p and T has in the final stage a p' and T' such that

$$p' = p_h + \frac{1}{l}\int_{l-x}^{l} (p_{0x} - p_h)\, dx$$

$$= p - \left(p - p_h - \frac{1}{l} \cdot \int_{l-x}^{l} (p_{0x} - p_h)\, dx \right)$$

In this last equation we consider the member in the parenthesis as small relative to p, which is true when the difference between any two values of the pressure within the mass is small as compared with the total pressure p_h. Under this assumption we introduce an approximate computation analogous to that of the last section § (25),

$$T - T' = T - T\left(\frac{p'}{p}\right)^{\kappa} = \kappa \left[T - \frac{p_h}{R\mu} - \frac{1}{lR\mu}\int_{l-x}^{l} (p_{0x} - p_h)\, dx \right].$$

We first seek for the mass-integral of $(T - T')$ throughout the whole mass of the unit column above the point x on the axis of abscissæ; in accord with the previous definition we put T_x^* for the average temperature of this column, then we have

$$T_x^* \frac{p_{0x} - p_h}{g} = T_x^* \int_0^h \mu dz = \int_0^h T \mu dz$$

whence we obtain the mass-integral of $(T - T')$ as follows:

$$\int_0^h (T - T')\,\mu dz = \frac{\kappa}{g}\cdot\left\{ T_x^*\,(p_{0x} - p_h) - \right.$$
$$\left. - \frac{gh}{R}\left[p_h + \frac{1}{l}\int_{l-x}^{l} (p_{0x} - p_h)\,dx \right] \right\} \quad \left.\right\} \ldots (A)$$

This expression multiplied by the factor $C_p dx$ and integrated throughout the whole length l gives the available kinetic energy of the whole system. But since we take the true average temperature T^* instead of the barometric average temperature therefore we first substitute

$$p_{0x} - p_h = p_h\left[\frac{gh}{RT_x^*} + \frac{1}{2}\left(\frac{gh}{RT_x^*}\right)^2 \right] \quad . \quad . \quad . \quad . \quad (B)$$

and remark that in the first member, on the right hand side of the integral (A), both terms of (B) as the serial development of $p_{0x} - p_h$ are to be used, but in the last member of (A) only the linear term in h or the first term of (B) need be considered, if we desire to go only as far in the result as terms of the order

$$\mathrm{T}^*\left(\frac{gh}{RT^*}\right)^2$$

For T_x^* we choose a linear function of the length in which τ is small relative to unity so that

$$\mathrm{T}_x^* = \mathrm{T}_0^*\left(1 + \tau\frac{x}{l}\right) \quad ; \quad \frac{1}{\mathrm{T}_x^*} = \frac{1}{\mathrm{T}_0^*}\left(1 - \tau\frac{x}{l}\right)$$

This gives for the last term in the integral (A)

$$\frac{1}{l}\int_{l-x}^{l} (p_{0x} - p_h)\,dx = p_h\,\frac{gh}{RT_0^*}\left(\frac{x}{l} - \tau\,\frac{2lx - x^2}{l^2}\right)$$

and for the complete integral (A)

$$\int_0^h (T - T')\,\mu dz = \frac{\kappa}{g} p_h\,\mathrm{T}_0^*\left(\frac{gh}{RT_0^*}\right)^2\left\{ \frac{1}{2} - \frac{x}{l} + \frac{\tau}{2}\cdot\frac{x}{l} - \frac{\tau x^2}{2l^2} \right\}$$

whence the available energy

$$\left.\begin{aligned} C_p \int (T - T')\, dm &= C_p \int_0^l dx \int_0^h (T - T')\, \mu\, dz \\ &= \frac{M}{2} V^2 = l p_h \frac{gh}{RT_0^*} \cdot h \cdot \frac{\tau}{12} \end{aligned}\right\} \quad . . (C)$$

Using the approximate value

$$M = l\, p_h \frac{gh}{R T_0^*}$$

we obtain

$$V = \sqrt{\frac{gh\tau}{6}} \quad . . . . . \text{(II)}$$

In this case the available kinetic energy for the unit of mass is smaller in the ratio of 2 to 3 than in the system previously considered, if we substitute T_0^* and T_1^* therein instead of T_1^* and T_2^*. In this first approximation, this energy depends only on the altitude and on the maximum horizontal difference of temperature and is independent of the length of the trough and therefore also of the horizontal temperature gradient.

The result is principally determined by the assumption that the vertical diminution of temperature is that corresponding to neutral equilibrium, as is shown in the following section.

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§(28) *To find the location of the surfaces of equal entropy in a mass of air when the pressure is constant throughout any level surface and the temperature is a function of the length and a linear function of the altitude.*

In the expression for the entropy

$$S = \text{Constant} + C_p \log T - R \log p,$$

we put

$$T = T_h + \frac{g}{n C_p}(h - z) \quad \text{and} \quad p = p_h \left(\frac{T}{T_h}\right)^{\frac{n C_p}{R}}$$

where T_h is a function of x only and p_h is constant. The system of curves of equal entropy in the xz plane is determined by the equation

$$F(x, z) = n \log T_h - (n - 1) \log T = \text{Constant}$$

The angle α that is included between the direction of the curve in the vertical plane of xz and the horizon is given by the equation

$$\tan \alpha = \frac{\partial F}{\partial x} \Big/ \frac{\partial F}{\partial z} = \frac{n}{n-1}\left(\frac{h-z}{T_h} + \frac{C_p}{g}\right)\frac{\partial T_h}{\partial x}$$

For $n=1$ the angle α is a right angle. If $n>1$ with stable equilibrium in each isolated column, and if x increases toward the right hand, then the curves of equal entropy trend downward toward the right. With increasing values of n the inclinations of the surfaces of equal entropy to the level surfaces diminish very rapidly.

If the horizontal increase of temperature is 1° C. for 20 kilometers or

$$\frac{\partial T_h}{\partial x} = 0.00005° \text{ Centigrade per meter}$$

we obtain the following respective sets of values:

n	Vertical Temp. Gradient per 100m	α_h for $z=h$	α_0 for $z=0$ $h=6000m$ $T_h=250°$
1.00	0.993	90° 0′	90° 0′
1.01	0.984	26° 57′	32° 12′
1.10	0.903	3° 10′	3° 55′
1.50	0.662	0° 52′	1° 4′
2.00	0.497	0° 34′	0° 43′

Even for very large horizontal temperature gradients, the surfaces of equal entropy are but slightly inclined to the level surfaces when the vertical diminution of temperature in dry air is less than 0.9° per 100 meters. If now from such an initial condition the masses pass adiabatically to the final stage then there is evidently available a smaller amount of kinetic energy than in the cases where the entropy surfaces are at first vertical.

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§(29) We now return to the two-chamber system.

To find the final stage of two masses of air under constant pressure with initial linear vertical diminution of temperature.

If the vertical gradient of temperature within the masses 1 and 2 (fig. 4) is smaller than that for neutral equilibrium then the entropy

increases with the altitude and it can happen that the entropy at the altitude c_1 in the cooler mass 1 is as large as that at the base of the mass 2, i. e., at the altitude $(h - c_2)$; the upper layers of 1 may one after the other serially have respectively the same entropy as the layers in 2 up to the altitude $(h - c_2)$. In the final stage the lower part of 1 will become spread out at the base; over it will lie strata consisting of 1 and 2 mixed; above this will rest that portion of 2 that initially lay between $(h - c_2)$ and h. At the boundaries between these three layers the temperature changes are continuous.

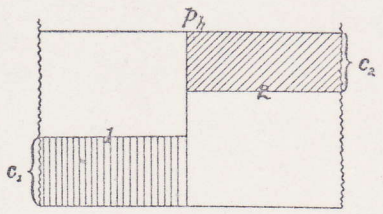

FIG. 4.

If the temperatures diminish linearly as in the equations

$$T_1 = T_{h1} + \frac{g}{n_1 C_p}(h - z) \qquad T_2 = T_{h2} + \frac{g}{n_2 C_p}(h - z)$$

then we have to seek the altitude at which the entropies are equal or $S_1 = S_2$ for the same value of p_h in the equations

$$S_1 = K + C_p[n_1 \log T_{h1} - (n_1 - 1) \log T_1]$$
$$S_2 = K + C_p[n_2 \log T_{h2} - (n_2 - 1) \log T_2]$$

Let us assume $n_1 = n_2 = n$; let θ_1 be the temperature of the mass 1 at the altitude c_1 and θ_2 the temperature of the mass 2 at the altitude $h - c_2$ then we have

$$\log \frac{T_{02}}{\theta_1} = \log \frac{\theta_2}{T_{1h}} = \frac{n}{n-1} \log \frac{T_{h2}}{T_{h1}}$$

$$c_1 = \frac{C_p}{n g}(T_{01} - \theta_1) \qquad c_2 = \frac{C_p}{n g}(\theta_2 - T_{h2})$$

Hence for $n = 2$ and a vertical temperature gradient of about 0.5° per 100 meters and for $h = 3000$ meters $T_{h1} = 263°$, $T_{h2} = 273°$, we find $c_1 = 2154$ meters and $c_2 = 2090$ meters.

In this case the greater part of the masses 1 and 2 remain unmixed, the available kinetic energy will not be much smaller than if in the final stage the whole of mass 1 lies below and the whole of 2 above.

Again, for $h = 6000$ meters $T_{h1} = 248°$, $T_{h2} = 258°$, we find $c_1 = 2390$ meters and $c_2 = 2096$ meters.

In this case the descent of the mass 1, and the ascent of the other mass 2, are much smaller than for the same values of h in the first and second analyses §(24) and §(25); hence also the available kinetic energy cannot be evaluated by using the equations there deduced.

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§(30) *Two masses of air having constant temperatures.* In order to be able to express more accurately the influence of the formation of three strata on $-\delta(\bar{P} + \bar{I})$ I have computed an example for the case of constant T_1 and T_2. In this case we have

$$S_1 = K + C_p \log T_1 - g\frac{h-z}{T_1}$$

$$S_2 = K + C_p \log T_2 - g\frac{h-z}{T_2}$$

$$\frac{gh}{T_2} - \frac{g(h-c_1)}{T_1} = \frac{gc_2}{T_2} = C_p \log \frac{T_2}{T_1}$$

In the final stage the temperature of the median layer is constant, viz.,

$$(T') = 2^{-\kappa}\left(T_1^{1/\kappa} + T_2^{1/\kappa}\right)^{\kappa}$$

For $h = 3000$ meters, $T_1 = 258°$, $T_2 = 268°$, we find $c_1 = 1099.5$ meters $c_2 = 1025.9$ meters, $(T') = 263.12°$.

The available kinetic energy is to be found as in the first computation but by a much longer route. The velocity for the average kinetic energy and for two chambers of equal volume is $V = 9.5$ meters per second or only half as much as for the more complete overflow from 2 and underflow from 1 as computed by equation (I) of section 25.

FOURTH ANALYSIS. EQUALIZATION OF THE PRESSURES IN A HORIZONTAL LAYER THAT RETAINS THE CONSTANT VOLUME

§(31) *Computation of the available kinetic energy.*

Initial stage. Assume a thin horizontal layer, bounded by rigid walls, divided by a screen into two chambers having volumes k_1 and k_2, whose masses are under the pressures p_1 and p_2 respectively. After removal of the dividing screen the pressure throughout the whole volume $k = k_1 + k_2$ becomes p'; if the expansion of one mass

and the compression of the other takes place adiabatically then the new volumes k'_1 and k'_2 will be

$$k'_1 = k_1\left(\frac{p_1}{p'}\right)^{1/\gamma} \qquad k'_2 = k_2\left(\frac{p_2}{p'}\right)^{1/\gamma}$$

From the condition that $k'_1 + k'_2 = k$ we find

$$p' = \left(\frac{k_1}{k}p_1^{\ 1/\gamma} + \frac{k_2}{k}p_2^{\ 1/\gamma}\right)^{\gamma}$$

The center of gravity of the whole mass remains at the same altitude wherefore $\delta\overline{P} = 0$.

The available kinetic energy is therefore

$$-\delta\overline{I} = (\overline{I_a}) - (\overline{I_e}) = C_v\int (T - T')\,\mu dk = \frac{C_v}{R}\int (p - p')\,dk$$

$$= \frac{1}{\gamma - 1}(k_1 p_1 + k_2 p_2 - k p')$$

If the difference $p_1 - p_2$ is small as compared with the pressure in one of the chambers so that $p_1 - p_2 = \varepsilon p_1$ then

$$p_2 = p_1(1 - \varepsilon)$$

$$p' = p_1\left(1 - \frac{k_2}{k}\varepsilon - \frac{k_1 k_2}{k^2}\cdot\frac{\gamma - 1}{\gamma}\cdot\frac{\varepsilon^2}{2}\right)$$

$$-\delta\overline{I} = \frac{1}{\gamma}\cdot\frac{k_1 k_2}{k^2}\cdot k p_1\cdot\frac{\varepsilon^2}{2}\cdot = \tfrac{1}{2}m\,(V)^2$$

Therefore if T^* is the average temperature of the whole mass, then the average available kinetic energy for the unit mass $\frac{p_1 k}{RT}$ is given by

$$\frac{[V^2]}{2} = \frac{(V)^2}{2} = \frac{k_1 k_2}{k^2}\cdot\frac{RT^*}{\gamma}\cdot\frac{\varepsilon^2}{2}$$

and when the other circumstances are the same this has its maximum when

$$k_1 = k_2 = \frac{1}{2}k.$$

If then the volumes of the chambers 1 and 2 are equal, we have

$$(V) = \frac{1}{2}\varepsilon.\sqrt{\frac{C_v}{C_p}RT^*} = \frac{1}{2}\frac{p_1 - p_2}{p_1}\sqrt{\frac{C_v}{C_p}R.T^*} \quad . \quad . \quad . \text{(III)}$$

If $p_1 = 765$ mm mercury, $p_2 = 755$mm mercury, $T = 273°$ then this equation gives $(V) = 1.55$ meters per second.

We will compare this value of (V) with the V that was deduced previously in §(3) for two masses that initially lay in a deep trough alongside of each other (see fig. 1). If h is the altitude [of the two chambers] and T_1^* T_2^* are the average temperatures and if the greatest entropy in mass (1) is smaller than that of the lowest layer of mass (2), then for chambers of equal volume the second analysis gave the velocity $V = \frac{1}{2}\sqrt{gh\tau}$ (see the expression (I) §25). This may be brought into a form similar to that given above in (III) if we put

$$\frac{gh}{RT_1^*} = \log\frac{p_{01}}{p_h} \qquad \frac{gh}{RT_2^*} = \log\frac{p_{02}}{p_h}$$

$$gh\tau = gh\frac{T_2^* - T_1^*}{T^*} = \text{approximately } RT^*\frac{p_{01} - p_{02}}{p_{01}}$$

whence

$$V = \frac{1}{2}\sqrt{\frac{p_{01} - p_{02}}{p_{01}}}\sqrt{RT^*} \quad . \quad . \quad . \quad . \quad . \quad . \quad (I^*)$$

If for p_{01} p_{02} and T^* we assume the same values as those just given $p_1 = 765$, $p_2 = 755$, $T = 273°$ there results $V = 16$ meters per second, or a velocity ten times larger than from equation (III), and therefore a hundred times the kinetic energy per unit mass.

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§(32) We can also add the following problem:

Let the chambers 1 and 2 of fig. 1 be limited above by a rigid partition and contain masses of air that have equal entropies but different pressures at the same altitudes, consequently there must be a higher temperature on the side of the higher pressure. The difference of pressure at any level is in the same direction at all altitudes and is nearly proportional to the average pressure at that level. The initial stages of 1 and 2 are respectively that of stable equilibrium. After removing the separating vertical partition the adjacent layers on the same level unite. The altitude of

the center of gravity remains unchanged. The resulting (V) has the same form (eq. III) as for the thin horizontal layer considered above, provided we now let T indicate the average temperature of the whole mass and let p_1 and p_2 indicate the initial values of the pressures at the base.

Again let there be resting in these chambers masses whose distribution of entropies is such as was assumed in the second analysis, see section 25; this equation (I) or (I*) applies to their overturning even in this present case of constant volume. For the same differences of pressure at the base, and for smaller differences above, and when $p_{01} - p_{02}$ is a small fraction, as is the case in our atmosphere, equation (I) gives a much larger value of the living force than equation (III).

It seems now to have been abundantly demonstrated that the available kinetic energy of such a system is not dependent materially on the horizontal differences of pressure but on the distribution of entropy and the buoyancy dependent thereon.

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§(33) *Appendix to the fourth analysis. Study of Joule's experiment relative to the mutual independence of the internal energy and the volume of a gas.*

In the first chamber of fig. 1 let the horizontal layer of gas be under the pressure p_1 but let the other chamber be empty or $p_2 = 0$; we now have the same arrangement as in Joule's experiment, if we put $k_1 = \frac{1}{2}k$.

For this case the first equations of our fourth analysis, § 31, become

$$p' = 2^{-\gamma} p_1$$

$$-\delta \bar{I} = \frac{C_v}{R}\left(\frac{k}{2} p_1 - k.2^{-\gamma} p_1\right)$$

$$= \frac{C_v}{R}\frac{kp_1}{2}(1-2^{1-\gamma})$$

If T is the initial temperature in chamber one, then the mass of the gas is

$$M = \frac{k}{2}\frac{p_1}{RT}$$

whence

$$-\frac{\delta \bar{I}}{M} = \frac{(V)^2}{2} = C_v T\,(1 - 2^{1-\gamma})$$

In the case of motion without friction we have for air

$$(V) = \sqrt{2\, C_v T\, (1 - 2^{-0.41})} = 0.7034\, \sqrt{C_v T}.$$

For the temperature $T = 273°$ this equation gives $(V) = 307 m./sec.$

If we assume that the kinetic energy is confined to that half of the mass that flows out of the first chamber into the second, we find the velocity 435 *m./sec.* These values are of the same magnitude as the molecular velocities. In small vessels the kinetic energy is very soon converted (by friction) into heat and is added to the internal energy of the system This added heat that we fail to notice in our atmospheric movements is decisive for the result of the laboratory experiment. Probably the experiment of Joule would have given a very different conclusion if he could have performed it in vessels of very large horizontal extent. In that case one should have observed a diminution of the internal energy at whose cost arose the kinetic energy of the systematic motion appropriate to the extent of the vessel. The previous portion of this article has been suggested by the relation of our problem to Joule's experiment but the following suggestion which has perhaps already been made elsewhere may be added. In the case when the volumes of the gas chamber and the vacuum chamber are unequal and with the assumptions that the motion is frictionless and that the change of condition is adiabatic and that at certain moments the pressure p' is uniform throughout the whole space, we have

$$p' = \left(\frac{k_1}{k}\right)^{\gamma} p_1$$

$$-\delta \bar{\mathrm{I}} = \frac{M\,(V)^2}{2} = M\, C_v T \left[1 - \left(\frac{k_1}{k}\right)^{\gamma - 1}\right]$$

If the vacuum chamber is very large relatively to the gas chamber so that $k_1/k = 0$ approximately and since $\gamma - 1 > 0$ therefore this equation gives

$$\frac{1}{2}\,(V)^2 = C_v T$$

On the other hand, from the kinetic theory of gases, if (u^2) is the mean of the squares of the velocities of the molecules of gas in the non-systematic free motions of molecules [or (u^2) is the square of the mean free path] we have

$$\frac{1}{2}\,(u^2) = \frac{3}{2} RT$$

Hence if $C_v = \frac{3}{2}R$ or $\gamma = \frac{5}{3}$ we have $(V) = (u)$. This is the value of γ that is found for non-atomic gases but for other gases γ is less than $\frac{5}{3}$ and (V) is greater than (u).

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§(34) *Comparison of the problem above treated with analogous analyses for incompressible liquids.*

We will now consider a system of liquids each of which retains its constant density during its change of position. The available kinetic energy is to be computed by the equation

$$\partial\overline{\mathrm{K}} + (\overline{\mathrm{R}}) = -\partial\overline{\mathrm{P}}$$

The potential energy of position for a unit column of a homogeneous body between the altitudes λ_{I} and λ_{II} is given by

$$[\mathrm{P}] = \int_{z_{\mathrm{I}}}^{z_{\mathrm{II}}} g z \mu dz = g\mu \frac{z_{\mathrm{II}}^2 - z_{\mathrm{I}}^2}{2}$$

(A) Let the chamber 1 (fig. 1) contain liquid of the constant density μ_1, the chamber 2 a liquid of smaller density μ_2. So long as the screen separates the two chambers there is neutral equilibrium on both sides of it. The volumes 1 and 2 are in this case assumed constant and therefore equal to 1′ and 2′ respectively, B being the area of either base. We now have for the initial and final stages

$$\overline{\mathrm{P}}_{1a} = \frac{B}{2} g\mu_1 \frac{h^2}{2} \qquad \overline{\mathrm{P}}_{2a} = \frac{B}{2} g\mu_2 \frac{h^2}{2}$$

$$\overline{\mathrm{P}}_{1e} = Bg\mu_1 \frac{h^2}{8} \qquad \overline{\mathrm{P}}_{2e} = Bg\mu_2 \frac{3h^2}{8}$$

After the removal of the vertical screen the available kinetic energy becomes

$$-\partial\overline{\mathrm{P}} = \frac{MV^2}{2} = gBh^2 \frac{\mu_1 - \mu_2}{8}$$

and the mass is

$$M = B\frac{\mu_1 + \mu_2}{2}$$

Let

$$\mu_1 = \mu\left(1 + \frac{\sigma}{2}\right) \qquad \mu_2 = \mu\left(1 - \frac{\sigma}{2}\right)$$

$$p_{01} = gh\mu_1 \qquad p_{02} = gh\mu_2 \qquad p_0 = gh\mu$$

then analogous to equation (I) see §25, or (I*) see §31, we have

$$V = \frac{1}{2}\sqrt{gh\sigma} \qquad \text{or} \qquad V = \frac{1}{2}\sqrt{gh \cdot \frac{p_{01} - p_{02}}{p_0}}$$

(B) Following the initial condition assumed in the third analysis we now assume that a parallelopipedon having the height h, length l and breadth unity, is filled with liquid whose density is a function of the length x only. We also assume that the density diminishes continuously from $x = 0$ up to $x = l$.

We thus have for the initial stage

$$\overline{P}_a = \int_0^l dx \int_0^h gz\mu dz = \frac{gh^2}{2}\int_0^l \mu dx.$$

In the final stage the liquid that was initially in a vertical column at x above the elementary strip dx with density μ becomes a horizontal layer at the altitude ζ with the thickness $d\zeta$.

Since $hdx = ld\zeta$ and $hx = l\zeta$ therefore for the final stage we have

$$\overline{P}_e = l\int_0^h g\mu\zeta d\zeta = \frac{gh^2}{l}\int_0^l \mu x\, dx$$

If now we put

$$\mu = \mu_0\left(1 - \sigma\frac{x}{l}\right)$$

there results

$$M = lh\mu_0\,(1 - \tfrac{1}{2}\sigma)$$

$$\overline{P}_a = gh^2 l\mu_0\left(\frac{1}{2} - \frac{\sigma}{4}\right) \qquad \overline{P}_e = gh^2 l\mu_0\left(\frac{1}{2} - \frac{\sigma}{3}\right)$$

$$-\delta\overline{P} = \overline{P}_a - \overline{P}_e = gh^2 l\mu_0\frac{\sigma}{12} = \frac{MV^2}{2}$$

and when σ is a small fraction then, and in agreement with (II) (§27) we have

$$V = \sqrt{\frac{gh\sigma}{6}}$$

(C) Let us now arrange the liquid in the trough in parallel inclined layers of equal density and assume for the initial stage

$$\mu = \mu_0\left(1 - \sigma_x\frac{x}{l} - \sigma_z\frac{z}{h}\right) = \mu_0\,(1 - \varphi).$$

The computation of the available kinetic energy for this case supplements the computation for air whose entropy is a function of length and altitude. The angular inclination to the horizon of the layers of equal density is

$$\alpha = \text{arc tan}\left(\frac{\sigma_x}{\sigma_z} \cdot \frac{h}{l}\right).$$

If $\sigma_x > \sigma_z$ then the layer that starts from the edge where $x = 0$ and $z = h$ will intersect the bottom of the trough.

The analysis must be executed separately for the three regions in which $\varphi = \sigma_x \frac{x}{l} + \sigma_z \frac{z}{h}$ is between 0 and σ_z; or between σ_z and σ_x; or between σ_x and $\sigma_x + \sigma_z$. The first and last of these regions are triangles in the xz plane, the second is a parallelogram. The evaluation of $\overline{P}_a$ and $\overline{P}_e$ for the separate regions is rather tedious. Eventually we find

$$V^2\left(1 - \frac{\sigma_x}{2} - \frac{\sigma_z}{2}\right) = gh\left\{\frac{\sigma_x - \sigma_z}{6} + \frac{1}{12} \cdot \frac{\sigma_z^2}{\sigma_x} \cdot \left(1 - \frac{1}{5} \cdot \frac{\sigma_z}{\sigma_x}\right)\right\}$$

If $\sigma_x < \sigma_z$ then the above described layer (that starts from the edge for which $x = u$, $z = h$) will intersect the opposite vertical wall of the trough. For this case we have

$$V^2\left(1 - \frac{\sigma_x}{2} - \frac{\sigma_z}{2}\right) = gh \cdot \frac{1}{12} \cdot \frac{\sigma_z^2}{\sigma_x} \cdot \left(1 - \frac{1}{5} \cdot \frac{\sigma_x}{\sigma_z}\right).$$

In this case also the available kinetic energy is independent of the length of the trough.

If $\sigma_z = 0$ [then $\alpha = 90°$ and the layers are vertical columns as in case B and] the former of these two equations becomes $V^2\left(1 - \frac{\sigma_x}{2}\right) = gh\,\frac{\sigma_x}{6}$ which is identical with the value of V computed in case B when σ_x is a small fraction.

If $\sigma_z = \sigma_x$ then the layers are parallel to the diagonal surface of the trough. For this case the two equations agree in giving

$$V^2(1 - \sigma) = \frac{gh\sigma}{15};$$

therefore, if σ is a small fraction, the available kinetic energy is about one-fourth of that which we found for the same value of σ in case (A) or two-fifths of the analogous quantity in case (B).

(D) Finally, in order to imitate with incompressible liquid the case treated in the fourth analysis we return to the two chambers; we assume their basal areas to be equal; the chamber 1 to be filled with liquid with density μ to the height $h + \frac{\eta}{2}$ and the chamber 2 filled with the same liquid to the height $h - \frac{\eta}{2}$, so that in the final stage the fluid extends to the altitude h throughout the whole trough. We now have

$$\overline{P}_a = \frac{B}{2} g\mu \left(h^2 + \frac{\eta^2}{4} \right) \qquad \overline{P}_e = \frac{B}{2} g\mu h^2$$

$$p_{01} = g\mu \left(h + \frac{\eta}{2} \right) \qquad p_{02} = g\mu \left(h - \frac{\eta}{2} \right) \qquad p_0 = g\mu h.$$

$$(V)^2 = \frac{g\eta^2}{4h} \qquad (V) = \frac{1}{2} \frac{p_{01} - p_{02}}{p_0} \sqrt{gh}$$

This last expression is the analogue of equation (III) of the fourth analysis, §31. For equal values of p_{01} and p_{02} and when their difference is small relative to p_0 then in this case D, the velocity (V), is much smaller than the V in case (A).

Chapter IV

The equation of energy for moist air in which condensation occurs in connection with the change of location

In order to investigate the influence of the latent heat of condensation on the available kinetic energy a fictitious gas is introduced.

§(35) We may consider the equation (6*) deduced previously for an ideal gas of constant composition as applicable to any closed system. The significance of $\overline{I}$ depends on the nature of the system. We will apply the equation of energy to an atmosphere composed of air, water, and vapor, assuming thereby that the vapor is an ideal gas up to its point of condensation. In this case we can deduce the equation of energy by a special method if we change the equation of continuity so as to include the processes of condensation and evaporation.

We adopt the equation (6*) of §11 as an axiom

$$(Q) = \delta (\overline{K} + \overline{P} + \overline{I}) + (\overline{R})$$

Of the many processes that occur in moist air we will base our analysis on those only that we consider important for our problem, which is to determine the available kinetic energy of any initial stage; all other considerations we temporarily omit.

The condensation of vapor into small drops causes the capillary energy to be included in $\bar{I}$, and the pressure of the saturated vapor to depend on the size of the drops, and the water floating in the air to contribute slightly by its weight to the pressure of the strata beneath. We free ourselves from these influences which are of minor importance in our present problem, by assuming that in our system the condensed water immediately falls to the ground [or at least separates from the air under consideration].

Let α, β, γ be the subscript indices referring to the air, vapor, and water, respectively, so that the mass (m) of moist air is composed of three portions: m_α; m_β; m_γ respectively; the specific heats are

$$C_{v\alpha}\ C_{v\beta}\ C_{v\gamma} : C_{p\alpha}\ C_{p\beta}\ C_{p\gamma} : \text{moreover } C_{p\beta} - C_{v\beta} = R_\beta, \text{ etc.}$$

L = latent heat of evaporation of a kilogram of water at the temperature T; and L' at the temperature T'.

$L - R_\beta T$ = internal latent heat of evaporation.

C = specific heat of water, assumed to be constant.

CT = internal energy of a kilogram of water.

$CT + L - R_\beta T$ = internal energy of a kilogram of vapor.

Assuming that aqueous vapor follows the laws of ideal gases we have

$$L = L_0 - (C - C_{p\beta})\ T$$

$$L - R_\beta T = L_0 - (C - C_{v\beta})\ T$$

$$L_0 = \text{Constant}$$

For the temperature T the internal energy $d\bar{I}$ of the elementary mass dm which is composed of dm_α, dm_β and dm_γ is given by the expression

$$d\bar{I} = C_{v\alpha}\ T\, dm_\alpha + C\, T\, (dm_\beta + dm_\gamma) + (L - R_\beta T)\, dm_\beta$$

$$= C_{v\alpha}\ T\, dm_\alpha + C_{v\beta}\ T\, dm_\beta + C\, T\, dm_\gamma + L_0\, dm_\beta$$

We will first assume that during any change in the system the constituents of any elementary mass remain the same, so that in the final stage all have again a common temperature as in the initial stage; therefore dm_α remains unchanged and $dm'_\beta + dm'_\gamma = dm_\beta + dm_\gamma$ where the superscript primes refer to the final stage.

§(35a) Let us consider a system that in its initial stage contains no liquid water. Its internal energy will be

$$\overline{\mathrm{I}}_a = C_{v\alpha}\int T\,d\,m_\alpha + C_{v\beta}\int T\,d\,m_\beta + L_0\int d\,m_\beta$$

For the final stage this becomes

$$\overline{\mathrm{I}}_e = C_{v\alpha}\int T'\,d\,m_\alpha + C_{v\beta}\int T'\,d\,m'_\beta + C\int T'\,d\,m'_\gamma + L_0\int d\,m'_\beta$$

Since we assume $dm_\gamma = 0$ therefore $dm'_\beta = d\,m_\beta - d\,m'_\gamma$ and

$$\overline{\mathrm{I}}_e = C_{v\alpha}\int T'\,d\,m_\alpha + C_{v\beta}\int T'\,d\,m_\beta + L_0\int d\,m_\beta - \\ - \int [L_0 - (C - C_{v\beta})\,T']\,d\,m'_\gamma.$$

The excess of the internal energy in the initial stage over that in the final stage is

$$-\delta\,\overline{\mathrm{I}} = \overline{\mathrm{I}}_a - \overline{\mathrm{I}}_e = C_{v_\alpha}\int (T - T')\,d\,m_\alpha + C_{v_\beta}\int (T - T')\,d\,m_\beta + \\ + \int (L' - R_\beta T')\,d\,m'_\gamma.$$

We assume further that the density of the water is infinitely great in comparison with the density of the air (or that the volume of the water is zero). Under this assumption $\overline{\mathrm{I}}_e$ and $\delta\overline{\mathrm{I}}$ retain their values when the condensed water falls to the ground.

If the system is, as before in figure 1, composed of a lower portion of the mass of atmosphere, extending up to the altitude h, and an upper portion that acts only as a piston of constant weight Bp_h, where B is the area of the base; if also the air is at rest in both its initial and its final stages, then for the potential energy of position we have the following expressions, in which the volume and mass integrals are to be extended over the lower portion only. This latter is true also for the mass-integrals in $\overline{\mathrm{I}}$. The partial pressures are p_α and p_β.

$$\overline{\mathrm{P}}_a = \int dB \int_0^h p\,dz + \text{Constant} = \int (p_\alpha + p_\beta)\,dk + \text{Constant}$$

$$= R_\alpha \int T\,dm_\alpha + R_\beta \int T\,dm_\beta + \text{Constant}$$

$$\overline{\mathrm{P}}_e = R_\alpha \int T'dm_\alpha + R_\beta \int T'dm'_\beta + \text{Constant}$$

$$= R_\alpha \int T'dm_\alpha + R_\beta \int T'dm_\beta - R_\beta \int T'dm_\gamma + \text{Constant}$$

The water that in the final stage is lying on the ground contributes nothing to the $\overline{P}_e$. The excess of the potential energy of the initial stage over the final stage is

$$-\delta\overline{P} = \overline{P}_a - \overline{P}_e = R_\alpha\int(T - T')\,dm_\alpha + R_\beta\int(T - T')\,dm_\beta + \\ + R_B\int T'dm'_\gamma$$

If we assume that each portion of the system passes adiabatically over to its final stage, then we have $(Q) = 0$ and the available kinetic energy is

$$\delta\overline{K} + (\overline{R}) = -\delta(\overline{P} + \overline{I}) = C_{p\alpha}\int(T - T')\,dm_\alpha + \\ + C_{p\beta}\int(T - T')\,dm_\beta + \int L'\,dm'_\gamma \qquad (F)$$

For moist air the final stage of stable equilibrium must be determined under conditions similar to those required for a gas of constant composition, i. e., that the strata be so arranged that the entropy increases upward. Assuming that we know the final location of each mass, then we can determine the pressure p' that the element dm' experiences in its final stage: since we also know the pressure p and temperature T for this element dm in its initial stage, therefore T' and thence L' and dm'_γ are to be computed from the well-known equations for the change of condition of moist air.

§(35*b*) We will now substitute a mixture of dry gases α and β whose composition has a local variability in place of the moist air [whose composition was uniform]. Assume that for each element of the mass the ratio $\frac{dm_\alpha}{dm_\beta}$ remains unchanged during the overturning.

The specific heat is determined by the relation

$$C_p\,dm = C_{p\alpha}\,dm_\alpha + C_{p\beta}dm_\beta$$

The change of the total potential energy due to the overturning process under constant pressure is

$$-\delta(\overline{P} + \overline{I}) = \int C_p(T - T')\,dm$$

$$= C_{p\alpha}\int(T - T')\,dm_\alpha + C_{p\beta}\int(T - T')\,dm_\beta$$

For the element dm the values of $(T - T')$ are the same as before

for moist air. But in order that this be possible, we must add heat during the overturning and the total quantity for the whole mass will be (Q). Hence according to (6*) the available kinetic energy will be

$$\delta \overline{\mathrm{K}} + (\overline{\mathrm{R}}) = (Q) + C_{p\alpha} \int (T - T')\, dm_\alpha + C_{p\beta} \int (T - T')\, dm_\beta$$

and the flow of heat is determined by the equation (F)

$$(Q) = \int L' dm'_\gamma \qquad (F^*)$$

It is to be noted that $L'dm'_\gamma$ is not exactly the latent heat of condensation that the element dm receives during its whole path, but the quantity of heat evolved by the condensation of dm'_γ at the temperature of the final stage of dm. This is in accordance with the assumption that was made in the computation of I_e where it was assumed that condensed water is carried along with the air to its final stage.

We will therefore now investigate the influence of the latent heat of condensation on the available kinetic energy, by means of another system that is more perspicacious than moist air. Since the local difference of composition is of slight influence in this problem we will replace the moist air by a homogenous gas that can expand with increase of heat.

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§(36) *Equations for a fictitious gas that receives increase of heat by its own expansion.*

The fictitious gas that we will introduce instead of moist air behaves when it is compressed, like dry air and obeys in general the equation of elasticity $p = RT\mu$. But with every diminution of pressure there is connected an addition of heat so that for expansion

$$\frac{dT}{T} = \lambda \frac{dp}{p} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where λ differs from the κ that holds good during compression. The quantity of heat imparted by this law of expansion is

$$dQ = C_p\, dT - \frac{RT}{p} dp = -(R - \lambda C_p)\, T \frac{dp}{p} \quad . \quad . \quad . \quad (2)$$

We assume that dQ is positive when dp is negative

therefore $R - \lambda C_p$ is positive

and also that $\lambda < \kappa$ . . . . . (3)

and that cooling accompanies expansion as λ is positive. . . (4)
and finally assume that λ is constant . . . . . (5)

If T_0 and p_0 belong to the initial stage of a mass then by the expansion or transition to a smaller pressure p we have

$$\frac{T}{T_0} = \left(\frac{p}{p_0}\right)^{\lambda} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1a)$$

$$[Q]_{p_0}^{p} = C_p \frac{\kappa - \lambda}{\lambda} (T_0 - T) \quad . \quad . \quad . \quad . \quad . \quad . \quad (2a)$$

Consider a vertical column filled with this gas at rest: Let p and T at the altitude z have the same values that a particle would have when ascending [adiabatically] from the base (p_0 and T_0) to this altitude.

The distribution of temperature in this column is now given by the above equation (1) by the condition of equilibrium (α) [§13], and by the equation of condition for elastic gases $p = RT\mu$. Combining these we obtain

$$\frac{1}{p}\frac{\partial p}{\partial z} = -\frac{g}{RT} = \frac{1}{\lambda} \cdot \frac{1}{T}\frac{\partial T}{\partial z}$$

whence

$$-\frac{\partial T}{\partial z} = \frac{g\lambda}{R} = \frac{\lambda}{\kappa} \cdot \frac{g}{C_p} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

With this fictitious gas, and for any given distribution of temperature in the vertical column, we can carry through a process similar to that considered in our preceding second chapter (§§ 13–22) and find that for ascending particles (or diminishing pressure) the diminution of temperature just given in equation (6) belongs to the condition of neutral equilibrium, and that a more rapid diminution of temperature corresponds to unstable equilibrium. In the first case the vertical temperature gradient is smaller than that for neutral equilibrium of dry air.

§(37) We now pass to the computation of the available kinetic energy of an extended horizontal system (fig. 1).

First analysis. Initial stage. Chamber 1 contains dry air, chamber 2 the fictitious gas; both of these are in neutral equilibrium, and both under the same pressure p_h at the altitude h. The temperatures are to be so chosen that, after the overturn, in the final stage, $2'$ lies wholly above $1'$ and p_h remains unchanged. Therefore if the serial sequence of the elementary layers be unchanged, every layer of chamber 2 expands except the highest one which retains its original pressure. The layers of $1'$ and $2'$ are individually in the condition of neutral equilibrium. As regards $2'$ this result follows from the application of equation (1*a*), §36.

Let (Q) be the quantity of heat added to the mass 2 by its expansion, then for the available kinetic energy of the whole system we have

$$\delta\overline{\mathbf{K}} + \overline{\mathbf{R}} = (Q) - \delta(\overline{\mathbf{P}} + \overline{\mathbf{I}}) = m\frac{(V)^2}{2}$$

$$= (Q) + C_p\left\{\int (T_1 - T_1')\,dm_1 + \int (T_2 - T_2')\,dm_2\right\}$$

The given data are h, p_h, T_{h1}, T_{h2}, λ, and β; whence for the initial stage we find

$$T_{01} = T_{h1}\left(1 + \kappa\frac{gh}{RT_{h1}}\right) \qquad p_{01} = p_h\left(1 + \kappa\frac{gh}{RT_{h1}}\right)^{\frac{1}{\kappa}}$$

$$T_{02} = T_{h2}\left(1 + \lambda\frac{gh}{RT_{h2}}\right) \qquad p_{02} = p_h\left(1 + \lambda\frac{gh}{RT_{h2}}\right)^{\frac{1}{\lambda}}$$

For the final stage we have

Locality	*Pressure*	*Temperature*
At the upper boundary of $2'$	p_h	T_{h2}
At the boundary i between the masses $2'$ and $1'$	$p_i' = \frac{1}{2}(p_{02} + p_h)$	$T_{h2}' = T_{h2}\left(\frac{p_i'}{p_h}\right)^{\lambda}$; $T_{h1}' = T_{h1}\left(\frac{p_i'}{p_h}\right)^{\kappa}$
At the base	$p_0' = \frac{1}{2}(p_{01} + p_{02})$	$T_0' = T_{h1}\left(\frac{p_0'}{p_h}\right)^{\kappa}$

By reason of equation (2a) we have

$$(Q) = C_p \frac{\kappa - \lambda}{\lambda} \int (T_2 - T_2')\, dm_2$$

whence

$$\delta \mathrm{K} + (\mathrm{R}) = C_p \int (T_1 - T_1')\, dm_1 + \frac{\kappa C_p}{\lambda} \int (T_2 - T_2')\, dm_2$$

In equation (5) in chapter II, §20, for a, the vertical gradient of temperature, substitute the values g/C_p for the masses 1 and 1′ and $\lambda g/(\kappa C_p)$ (see eq. 6 §36) for the masses 2 and 2′ and we obtain

$$\int T_1\, dm_1 = \frac{B}{2g} \cdot \frac{1}{1+\kappa} \cdot (p_{01} T_{01} - p_h T_{h1})$$

$$\int T_1'\, dm_1 = \frac{B}{g} \cdot \frac{1}{1+\kappa} \cdot (p_0' T_0' - p_i' T_{i1}')$$

$$\int T_2\, dm_2 = \frac{B}{2g} \cdot \frac{1}{1+\lambda} \cdot (p_{02} T_{02} - p_h T_{h2})$$

$$\int T_2'\, dm_2 = \frac{B}{g} \cdot \frac{1}{1+\lambda} \cdot (p_i' T_{i2}' - p_h T_{h2})$$

§(38) Example. In order to make the fictitious gas similar to moist air we compute the values for the initial condition in a different order of succession than that above given.

Initial stage. For mass 2: assume $T_{02} = 303°$, $p_{02} = 760$, $p_h = 500^{mm}$ mercury and seek first the value of T_{h2} for saturated moist air, that is to say, the temperature that such air attains when it expands from 303° and 760^{mm} to 500^{mm} mercury. This value lies between 289° and 290°.

We adopt $T_{h2} = 290°$ and with it by equation (1a) compute $\lambda = 0.1047307$ and further the vertical gradient of temperature in mass 2 or $a_2 = \frac{\lambda}{\kappa} \frac{g}{C_p} = 0.0035780$ degree Centigrade per meter which is nearly 3.6° per 1000 meters.

Finally we compute

$$h = \frac{T_{02} - T_{h2}}{a_2} = 3633.29 \text{ meters}$$

Initial stage. For mass 1 we adopt the same temperature at

the base T_{01} as for mass 2, i.e., 303°, whence for the same altitude $h = 3633.29$ meters we find T_{h1} and p_{01} as follows:

$$\left.\begin{matrix} T_{01} = 303^\circ \\ h = 3633.29 \end{matrix}\right\} T_{h1} = 266^\circ.9061 \qquad p_{01} = 773.4023^{mm} \text{ mercury.}$$

Final stage. For the final stage we compute the following values

$p'_i = 630^{mm}$ $\quad T'_{i2} = 297.1047^\circ$ $\quad T'_{i1} = 285.4593^\circ$

$p'_0 = 766.7011^{mm}$ $\quad T'_0 = 302.2342^\circ$

$h'_1 = 1688.60$ meters. $\quad h'_{i2} = 1985.65$ meters.

From these we now obtain

$$\int (T_1 - T'_1)\, dm_1 = -2.2312\, p_h \frac{B}{g}$$

$$\int (T_2 - T'_2)\, dm_2 = +2.3323\, p_h \frac{B}{g}$$

$$M \frac{V^2}{2} = C_p\, 0.1011\, p_h \cdot \frac{B}{g}$$

$$M = 0.5334\, p_h \frac{B}{g}$$

$$V = 19.3 \text{ meters per second.}$$

The true average temperatures of the masses 1 and 2 in the initial stage are $T^*_1 = 285.9^\circ$ and $T^*_2 = 296.9^\circ$.

If instead of the fictitious gas we had assumed dry air of the same average temperature in chamber 2 and if its entropy had been such that in the final stage 2′ it lay uppermost spread over 1′ then we should have computed the corresponding $V = 18.5$ meters per second from the approximate formula (I) §(25). Therefore the available energy is not much larger for the fictitious gas in chamber 2 than for dry air of the same temperature.

With the above given equation for (Q) (see 2*a*, §36), we compute the quantity of heat communicated, per unit area of the base of the whole trough, to the mass 2 during its expansion

$$\frac{(Q)}{B} = 2409 \frac{\text{Calories}}{M^2}$$

If this is due to precipitation of aqueous vapor then it corresponds to the condensation of 4 kilograms of water per square meter or to a depth of rainfall of 4^{mm} over the whole area.

If instead of the fictitious gas in chamber 2 we had assumed saturated moist air then we could have found the approximate quantity of water condensed and falling away from it as follows:

By the overturning the lowest layer of 2 changes from $T_{02} = 303°$ and $p_{02} = 760^{mm}$ to $T'_{i2} = 297$ and $p'_i = 630^{mm}$. In this lowest layer there is initially 0.02696 kg. vapor associated with each kilogram of dry air but in the final stage this is reduced to 0.02271; therefore in this layer there condenses 0.00425 kg. water out of every 1.027 kg. of mixed air and saturated vapor. The highest layer of mass 2 does not expand and contributes no water; we may assume that the whole mass 2 contributes on the average 0.0021 kg. water per kg. of its own mass so that the unit column gives up

$$\frac{260}{760} \times 10333.0 \times 0.0021 = 7.4 \text{ kg. of water}$$

Because of the spread of the mass 2 over the whole trough this water is distributed over double the original base giving 3.7^{mm} depth rainfall in close agreement with the value above given.

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§(39) *Second analysis. Approximate method for the computation of the available kinetic energy when the chamber 1 is filled with dry air and chamber 2 with the fictitious gas.*

In this case all the considerations that were made in the analogous second analysis, §25, chapter III, are to be repeated excepting only that λ is used instead of κ in chamber 2.

Initially the masses 1 and 2 are each in stable equilibrium within itself; the succession of layers within each mass remains unchanged in the overturning. Hence we obtain p'_1 and p'_2 and, assuming equal volumes for the two chambers, we find the following approximate values:

$$T'_1 = T_1\left(1 + \frac{\kappa}{2} \cdot \frac{p_{02} - p_1}{p_1}\right)$$

$$T'_2 = T_2\left(1 - \frac{\lambda}{2} \cdot \frac{p_2 - p_h}{p_2}\right)$$

$$\delta\overline{\mathrm{K}}+(\overline{\mathrm{R}})=C_p\left\{\int(T_1-T_1')\,dm_1+\frac{\kappa}{\lambda}\int(T_2-T_2')\,dm_2\right\}$$

$$=C_p\frac{B\kappa}{4g}\left\{\mathrm{T}_1\,(p_{01}-p_h)+\mathrm{T}_2\,(p_{02}-p_h)-\frac{gh}{R}\,(p_{02}+p_h)\right\}$$

This is the same value as before, and now the same considerations as in §25 lead to the approximate formula (I) of that article for the velocity V. We arrive at the same result as if instead of the fictitious gas in chamber 2 we had used dry air of the same average temperature. The reason why this happens is shown by the course of the analysis. The change of $\overline{\mathrm{P}}+\overline{\mathrm{I}}$ for the unit mass remains the same no matter whether chamber 2 is filled with dry air or with the gas; the amount that the mass 1 contributes to the available kinetic energy remains unchanged.

The contribution of mass 2 by simple change of location only, or the change of $\overline{\mathrm{P}}+\overline{\mathrm{I}}$ for this mass, $\left(\text{which is } C_p\int(T_2-T'_2)\,dm_2\right)$, is in the new case (for the fictitious gas) smaller than for dry air since the gas cools less by expansion and therefore T'_2 is larger.

But on account of the development of heat associated with the expansion (which we have introduced as equivalent to the latent heat of condensation) there is (Q) to be added to the expression $\delta\,\overline{\mathrm{K}}+(\overline{\mathrm{R}})$ and for the fictitious gas the expression

$$(Q)+C_p\int(T_2-T_2')\,dm_2$$

is as large as the second term alone would be for dry air.

Thus it is that the addition of heat by virtue of the expansion causes no increase in the available kinetic energy, but serves only to warm the expanding mass or to diminish its cooling.

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§(40) *The difference between the fictitious gas and the moist air.*

In order to simplify the analysis and investigate separately the influence of the latent heat of condensation we have given the fictitious gas that has replaced moist air the properties of dry air and have only introduced the condition that it shall expand when heat is added. In order that it might more nearly resemble moist air we should have also assumed its density smaller and its R to be variable with its condition.

The fictitious gas cannot replace moist air in every respect. The proof that the condensation has no influence on the available kinetic energy, does not exclude the possibility that the proximity of masses of relatively dry air and moist air should favor the origination of a storm.

Since the available kinetic energy of the system is derived from the buoyancy, therefore any diminution of density by reason of the vapor content of a mass of air operates like an increase of temperature. A particle of the fictitious gas remains in stable equilibrium in dry air, if it has the same temperature as the surrounding air and if the vertical temperature gradient in that air is smaller than $\lambda g/R$. A particle of saturated moist air, having the same exponential factor λ [would not be in stable or neutral equilibrium but unstable and] would rise because of its smaller density.

In the example of the first analysis §38, if instead of the fictitious gas we had used saturated moist air of the same temperature and pressure, then for the altitudes o and h its densities would have been respectively equal to that of dry air at $I_{02}=307.8°$ and $p_{02} = 760^{mm}$ mercury and that of dry air at $T_{0h}= 293.2°$ and $p_{oh}=500^{mm}$ mercury. This 307.8° corresponds to a difference of $T_{02} - T_{01}= 1.8°$ from the value $T_{01} = 303.0°$ as there adopted and a difference of 3.2° from the value $T_{h2}=290°$ there computed for the altitude h or a difference of 4.0° from the average temperature of the whole mass of 2 [when it is composed of saturated air instead of the fictitous gas]. Therefore for these two cases the available kinetic energy is in the ratio 15/11 and the velocities V are larger than in the fictitious gas in the ratio 1.17/1. Or, if we require the velocity V to be the same as before, then we need a smaller difference of temperature between dry and moist air, viz., about 8° instead of 11°. All this relates to the unusual high temperatures of our examples; for lower temperatures the influence of the moisture on the density will be still smaller.

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§(41) The kinetic energy of a mass of air is derived from its internal energy and from the work done by the force of gravity. In the case of a continuous distribution of density the importance of gravity in the production of great velocities can be concealed, whence we derive the very common belief that the horizontal gradient of pressure produces the storm. But it is now demonstrated that, even when the distribution of pressure at the base is as observed in storms still the horizontal movements of the masses

have a potential energy that is only a small fraction of the observed kinetic energy. So far as I can see the source of storms is to be sought only in the potential energy of position. A system in which the masses are disturbed vertically from equilibrium can contain the necessary potential energy. Hence, therefore, the storm winds develop by reason of the velocity due to descent and that due to buoyancy, notwithstanding the fact that these evade attention because of the large horizontal and small vertical dimensions of the storm area. The horizontal distribution of pressure appears as a translation of the driving power of the storm; by means of it a portion of the mass can attain greater velocity than by simple ascent in the coldest or descent in the warmest portion of the storm area. Here we come into the presence of problems that cannot be solved by a simple consideration of the energy alone.