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AND DIRAC'S CONJECTURE

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Buying a Station or Relaxing a Constraint? Spectrum Repacking and Dirac's Conjecture

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ABSTRACT

Spectrum repacking auctions can restore feasibility along two margins: retiring a station via buyout or relaxing a pairwise engineering restriction. I represent stations as vertices in a network and interference restrictions as edges between them, and so station buyouts delete vertices and engineering waivers relax edges. This paper shows that buying out any one station can reduce the number of channels needed, even though relaxing any single interference restriction cannot. The smallest feasible number of channels is k . For every $k \geq 4$, I show that there is a network in which removing any station saves one channel, while relaxing any single pairwise restriction does not. This main result follows directly from Dirac's half-century-old conjecture on vertex-critical graphs, whose chromatic number falls after any vertex is removed. By constructing the missing counterexample at $k=4$ and combining it with earlier results for $k \geq 5$, I show that at every $k \geq 4$, some k -vertex-critical graph has no critical edge, resolving Dirac's conjecture. Market designers auditing feasibility link-by-link systematically can mistake combinatorial complementarity for remedy failure, overlooking station buyouts that unlock the clearing target.

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1 Introduction

Spectrum auction is one of the major successes of modern market design. It buys broadcast rights from some television stations¹ and moves the stations that remain on air into a smaller part of the spectrum. The second task determines whether the auction succeeds in finding a feasible allocation. Stations that interfere with one another cannot use the same channel, so every remaining station must be assigned an available channel without causing harmful interference. This repacking problem was central to the design of the U.S. broadcast incentive auction ([Milgrom and Segal, 2020](#)).

In this paper, I bring this multi-billion-dollar market design problem into a simple model. This model represent each station as a vertex in an interference network and two stations are linked if they cannot use the same channel. Assigning channels naturally becomes equivalent to coloring the network, and its chromatic number is the fewest channels needed to keep every interfering pair apart. If the available channels fall short of that number, the regulator has two actions, corresponding to real-world policy levers, to make room and address interference. Buying a station removes that vertex and all the restrictions involving it, while an engineering intervention—such as costly electrical engineering or coordinating their operations—removes one restriction between a pair of stations, and this is modeled as deleting the corresponding edge from the network. The former actions removes a bundle of restrictions, whereas an engineering intervention relaxes only one. I show that, whenever at least four channels are initially required, there are networks in which buying any one station reduces the channel requirement by one but relaxing any one interference restriction does not. A market designer who examines restrictions one at a time can therefore overlook what a station purchase would accomplish.

To achieve her objectives, the market designer needs to evaluate how the structure of the network responds to these two different actions. One action removes every restriction while the other only one, and they do not make comparable changes to the network. Restrictions may matter together even when none matters by itself. The main economic theorem states that, for any initial requirement of k channels where $k \geq 4$, there is an interference network that requires all k channels but can fit onto $k - 1$ channels after any one station is removed but the same reduction remains impossible after any one interference restriction is relaxed. Every station is pivotal for the lower clearing target, even though no individual pairwise restriction is pivotal.

This economic theorem is a direct translation of a conjecture attributed to Dirac ([Dirac \(1952\)](#)). In graph-theoretical terms, the conjecture asks whether a network can be minimally demanding with respect to every station but robust to the loss of any one interference link. Earlier work proved this for every channel requirement of five or more; the four-channel case remained open ([Jensen, 2002](#); [Skottova and Steiner, 2025](#); [Ferudun, 2026](#)). I construct and verify the missing four-channel network, completing Dirac’s Conjecture.

¹There are other users other than TV stations but for simplicity of exposition, I will use TV stations or just stations to represent this broad set of potential users.

More broadly, this paper is an example of how AI-supported mathematics can advance economics, especially applied theory. Novel and less usual mathematical results or objects often shed on theoretical questions in economics and AI can help researchers identify promising candidates and enlighten on their nature to facilitate connecting abstract mathematics to economic institutions. Here, AI-assisted computation helped identify the graph that anchors the theory that enabled the results.

The paper is organized as follows. Section 2 states the spectrum model and result. Section 3 gives the construction and proof of Dirac’s Conjecture. Section 4 gives the policy and market design implications. Appendix A gives the verification details.

2 Spectrum repacking

2.1 Environment

Consider a finite set of broadcast stations. Represent their interference restrictions by a graph G : each station is a vertex, and an edge joins two stations when they cannot use the same channel. A channel assignment is *feasible* if stations joined by an edge receive different channels. The *channel requirement*, written $\chi(G)$, is the smallest number of channels that permits a feasible assignment. The model assumes that channels are interchangeable, that each station receives one channel, and that the same pairwise restrictions apply on every channel. It deliberately omits station-specific channel availability and other engineering details of an actual repacking problem.

For a station v , write $G - v$ for the network left after that station exits. This operation removes the station and every edge attached to it. For an interference restriction e , write $G - e$ for the network left after that one restriction is relaxed. All stations remain in $G - e$.

Let k be the number of channels the full network needs. The regulator wants the remaining stations to fit on $k - 1$ channels. We study integers $k \geq 4$. A station is *pivotal* for this target if its exit makes $k - 1$ channels sufficient. An interference restriction is pivotal if relaxing that restriction makes $k - 1$ channels sufficient.

2.2 Result

Theorem 1 (Pivotal stations and nonpivotal pairwise restrictions). *For every integer $k \geq 4$, there exists a spectrum-repacking environment such that*

1. *the full network requires k channels;*
2. *after any one station exits, the remaining network requires exactly $k - 1$ channels; and*
3. *after any one pairwise interference restriction is relaxed, the full set of stations still requires k channels.*

Proof. Corollary 2 proves that, for every integer $k \geq 4$, there is a graph whose channel requirement is k , whose requirement falls to $k - 1$ after any vertex is deleted, and whose requirement remains k

after any edge is deleted. Interpret vertices as stations, edges as pairwise interference restrictions, and colors as channels. The three graph properties are exactly statements 1–3. $\square$

Put plainly, the regulator may face a network in which buying *any* one station is enough to meet the clearing target, while fixing *any* one interference relationship is not enough. The relevant unit of flexibility is the station together with all of its restrictions, not a restriction considered alone.

The theorem describes feasibility only. It does not, by itself, determine bids, payments, or welfare. The next corollary states the decision implication that follows without adding a model of strategic behavior.

Corollary 1 (Least-cost exit). *Fix one of the environments in Theorem 1. Suppose the regulator can meet the target only by purchasing station exits, observes a nonnegative exit cost for each station, and minimizes total purchase cost. Then purchasing a station with the lowest exit cost is optimal.*

Proof. Theorem 1 says that every single-station exit meets the target. Any feasible purchase of at least one station therefore costs at least as much as a lowest-cost station, whose exit is itself feasible. $\square$

The corollary says only which observed-cost purchase is cheapest among the actions studied here. When costs are private, an auction may be used to learn which stations will sell and at what terms. Milgrom and Segal (2020) study clock procurement rules for the broader problem in which feasible sets must be found while prices fall. I do not characterize bidding, payments, or welfare.

Theorem 1 also rules out a tempting screen. A procedure that treats a station as useful only when at least one of its individual interference restrictions is pivotal rejects every station in these networks, although purchasing any station meets the target. Testing restrictions one at a time is therefore not a valid test of whether a whole station can create enough room.

3 Mathematical foundation

For any graph H , let $V(H)$ be its vertices and $E(H)$ its edges. A *proper q -coloring* assigns one of q colors to each vertex so that adjacent vertices receive different colors. The chromatic number $\chi(H)$ is the smallest such q . For $v \in V(H)$ and $e \in E(H)$, the graphs $H - v$ and $H - e$ are obtained by deleting the indicated vertex or edge. A graph is *k -vertex-critical* if $\chi(H) = k$ and $\chi(H - v) = k - 1$ for every vertex v . An edge is *critical* if deleting it lowers the chromatic number.

Dirac’s conjecture states that, for every integer $k \geq 4$, some k -vertex-critical graph has no critical edge. Jensen (2002) proved the claim for every $k \geq 5$. The construction below supplies the missing case $k = 4$.

3.1 The four-channel construction

I call the graph constructed below G^* . Its 60 vertices will be labeled 0 through 59, and n will denote the number of vertices in a candidate graph.

The proof rests on one simple idea. Remove vertex 0 and color the remaining graph with three colors. In every such coloring, each color appears on exactly two of the six neighbors of vertex 0. Restoring vertex 0 therefore creates two conflicts, whichever color it receives. Removing one edge can eliminate at most one of them. Symmetry then carries the argument from vertex 0 to every vertex and from its six incident edges to every edge.

I found G^* through an AI-assisted computational search. I launched a restriction-aware (based on previous work by [Ferudun \(2026\)](#)) search beyond small orders², deliberately attempting to use unconventional algebraic³ and geometric structures, which led to the 60-vertex Cayley graph below. OpenAI’s GPT-5.6 Sol model, accessed through ChatGPT, helped me run the search strategy and program that identified the candidate. The search served only to identify a candidate. The mathematical argument below, the two deterministic exhaustive Python verifiers, and the Lean 4 development described in [Appendix A](#) independently establish every property used in the paper.

For the exact construction, let $\mathbb{Z}_n$ denote the integers modulo the positive integer n , and let $\text{Aut}(\mathbb{Z}_5)$ be the group of automorphisms of $\mathbb{Z}_5$. Define the homomorphism $\varphi : \mathbb{Z}_{12} \rightarrow \text{Aut}(\mathbb{Z}_5)$ by $\varphi(a)(j) = 2^a j$. Because $2^{12} = 1 \pmod{5}$, this homomorphism is well defined; moreover, $\varphi(a+b) = \varphi(a)\varphi(b)$, so it gives the required group action. Let Γ be the semidirect product $\mathbb{Z}_5 \rtimes_{\varphi} \mathbb{Z}_{12}$, written with the $\mathbb{Z}_{12}$ coordinate first. Its elements are pairs $(a, i) \in \mathbb{Z}_{12} \times \mathbb{Z}_5$, with multiplication

$$(a, i)(b, j) = (a + b \pmod{12}, i + 2^a j \pmod{5}). \quad (1)$$

Thus Γ has 60 elements. Arithmetic in the first coordinate is modulo 12; arithmetic in the second coordinate, including powers and inverse powers of 2, is modulo 5. Define the six-element set

$$S = \{(1, 0), (11, 0), (5, 1), (7, 2), (6, 0), (6, 3)\}. \quad (2)$$

The set is closed under taking inverses: whenever s belongs to S , its group inverse also belongs to S .

The graph G^* has one vertex for each element of Γ . Two vertices g and h are adjacent when $h = gs$ for some $s \in S$. This is the right Cayley graph $\text{Cay}(\Gamma, S)$. Represent $a \in \mathbb{Z}_{12}$ by $0, \dots, 11$ and $i \in \mathbb{Z}_5$ by $0, \dots, 4$, and label (a, i) by

$$\ell(a, i) = 5a + i \in \{0, \dots, 59\}.$$

For a vertex r in a graph H , let $N_H(r)$ denote the neighbors of r ; write $N(r)$ when the graph is clear. In G^* ,

$$N(0) = \{5, 26, 30, 33, 37, 55\}. \quad (3)$$

²My approach requested the search to sidestep the fixation on finding the smallest graph near the $n \leq 15$ exclusion and led to a search to order 60 where enough symmetry and algebraic structure were available

³The successful graph is a Cayley graph on a specific nonabelian group, and the symmetry made vertex-criticality manageable as one vertex-deletion analysis could be transported to all 60 vertices, while edge behavior reduced to a few symmetry classes.

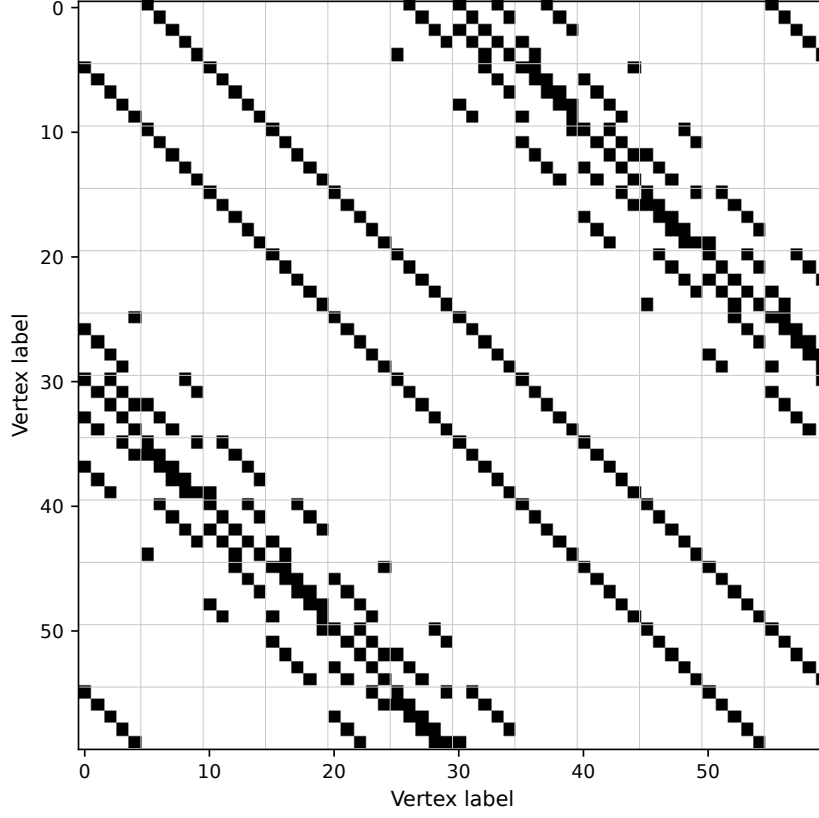


Figure 1: The interference matrix of G^* . Rows and columns represent the 60 stations; a dark off-diagonal cell marks an interference restriction. Each row contains six such cells. The image makes the network visible, while the proof uses the algebraic definition in equations (1)–(2) and the frozen data described in Appendix A.

Lemma 1 (Structure). *The graph G^* is a finite, simple, undirected graph with 60 vertices and 180 edges. Every vertex has six neighbors, and left multiplication by an element of Γ is a graph automorphism—a relabeling that preserves every edge—that can carry any vertex to any other vertex.*

Proof. The inverse of (a, i) is $(-a, -2^{-a}i)$. The six elements of S are distinct and are not the identity. Their inverse pairs are

$$(1, 0) \leftrightarrow (11, 0), \quad (5, 1) \leftrightarrow (7, 2), \quad (6, 0) \leftrightarrow (6, 0), \quad (6, 3) \leftrightarrow (6, 3).$$

The graph is therefore simple and undirected. Each of its 60 vertices has six neighbors, so it has $60 \cdot 6/2 = 180$ edges. Left multiplication preserves adjacency and acts transitively on the vertices. $\square$

The lemma says that all 60 stations occupy identical positions in the network. This symmetry lets one vertex-deletion calculation apply to every station.

Lemma 2 (Balanced puncture). *Let H be a finite graph, let $r \in V(H)$, and let $p \geq 2$ be an integer. Suppose $H - r$ has a proper p -coloring and, in every such coloring, each color appears on at least two vertices of $N_H(r)$. Then H is not p -colorable. Moreover, $H - e$ is not p -colorable for every edge $e = \{r, s\}$ with $s \in N_H(r)$.*

Proof. If H had a proper p -coloring, restricting it to $H - r$ would give one of the colorings in the premise. The color assigned to r would also appear on a neighbor of r , which is impossible in a proper coloring. Now suppose $H - e$ had a proper p -coloring. Restricting it to $H - r$ again invokes the premise. At least two neighbors of r have the color assigned to r , while deleting e removes at most one of the resulting conflicts. At least one undeleted monochromatic edge remains, a contradiction. $\square$

In every allowed coloring after r is removed, each color is used by at least two of r 's former neighbors. Bringing r back therefore creates at least two conflicts, so deleting one incident edge cannot make the coloring feasible.

To state the finite step, let $c(v) \in \{0, 1, 2\}$ denote the color assigned to vertex v . Vertices 1 and 6 are adjacent in $G^* - 0$, so their colors differ. Every proper three-coloring can therefore be renamed in one and only one way so that $c(1) = 0$ and $c(6) = 1$. Call this the *normalized* version of the coloring.

Lemma 3 (Finite enumeration). *The graph $G^* - 0$ has exactly 16 normalized proper three-colorings. In every one,*

$$(c(5), c(26), c(30), c(33), c(37), c(55)) = (1, 1, 0, 2, 2, 0). \quad (4)$$

Computer-assisted proof. The primary deterministic verifier described in Appendix A constructs the graph from equations (1)–(2) and checks it against the frozen edge list. It then fixes $c(1) = 0$ and $c(6) = 1$ and exhaustively branches over every permissible color at each remaining vertex. The search finds exactly 16 colorings and checks equation (4) in each. Appendix B displays all 16 through a compact template. A separately written exact verifier repeats the enumeration with a different search method. The Lean 4 development formalizes the balanced-puncture conclusion needed for Theorem 2; Appendix A states the role and shared inputs of each route. $\square$

The enumeration has a simple implication. After vertex 0 is removed, each of the three colors appears exactly twice among its six former neighbors. If vertex 0 returns, every possible color creates two conflicts. Deleting one of its six incident edges can remove only one conflict, so three channels are still insufficient.

Theorem 2 (The missing four-channel graph). *The graph G^* satisfies*

$$\chi(G^*) = 4, \quad \chi(G^* - v) = 3 \quad \text{for every } v \in V(G^*), \quad \chi(G^* - e) = 4 \quad \text{for every } e \in E(G^*).$$

Thus G^ is 4-vertex-critical and has no critical edge.*

Proof. Lemma 3 gives a proper three-coloring of $G^* - 0$. Giving vertex 0 a fourth color shows that $\chi(G^*) \leq 4$. Equation (4) says that each of the three colors occurs twice on $N(0)$ in every three-coloring of $G^* - 0$. Lemma 2 therefore gives $\chi(G^*) > 3$ and shows that deleting any edge incident to 0 does not make the graph three-colorable. Hence $\chi(G^*) = 4$.

By Lemma 1, a left translation carries 0 to any vertex v and transports a three-coloring of $G^* - 0$ to a three-coloring of $G^* - v$. No $G^* - v$ is two-colorable, because assigning a third color to v would then three-color G^* . Thus $\chi(G^* - v) = 3$ for every vertex v .

Finally, a left translation carries every edge to an edge incident to 0. The edge-deletion conclusion of Lemma 2 therefore applies to every edge. Since $G^* - e$ is a subgraph of the four-colorable graph G^* , $\chi(G^* - e) = 4$. $\square$

This network needs four channels. Removing any station makes three channels enough, but removing any single interference link does not.

For completeness, define the *three-color repair number* of a graph to be the fewest edges that must be removed to make the graph three-colorable.

Proposition 1 (Exact repair number). *The three-color repair number of G^* is two. In particular, each of the following pairs of edges is a repair:*

$$\{\{0, 5\}, \{0, 26\}\}, \quad \{\{0, 30\}, \{0, 55\}\}, \quad \{\{0, 33\}, \{0, 37\}\}.$$

Proof. Take any coloring from Lemma 3. Assigning color 1 to vertex 0 makes exactly the edges $\{0, 5\}$ and $\{0, 26\}$ monochromatic; deleting them gives a proper three-coloring. Assigning color 0 or 2 gives the other two displayed pairs. Thus two deletions suffice. Zero deletions cannot suffice because G^* needs four colors, and one cannot suffice because Theorem 2 says that no single edge is critical. $\square$

One relaxation never works in this particular network, but two carefully chosen relaxations do. This is a statement about feasibility only. It says nothing about the cost or welfare effect of either repair.

3.2 Completion of Dirac's conjecture

Adopt the standard convention that the graph with no vertices has chromatic number zero.

Corollary 2 (Dirac's conjecture and the sharp boundary). *For every integer $k \geq 4$, there exists a finite simple graph H such that*

$$\chi(H) = k, \quad \chi(H - v) = k - 1 \text{ for every } v \in V(H), \quad \chi(H - e) = k \text{ for every } e \in E(H).$$

No such graph exists for $k = 2$ or $k = 3$. At $k = 1$, the one-vertex graph satisfies the conditions only vacuously because it has no edge.

Proof. Theorem 2 proves the claim for $k = 4$. Jensen (2002) proves it for every integer $k \geq 5$; Skottova and Steiner (2025, Theorem 1.3, specialized to one-edge robustness) gives the same existence conclusion in a stronger form. Appendix C proves the claims for $k = 1, 2, 3$. $\square$

The case $k = 1$ is vacuous: its witness has no interference link to relax. Thus, among networks containing an actual interference restriction—and hence among channel requirements of at least two—four is the first level at which every station can be pivotal while every individual link is nonpivotal. With two or three channels, that separation is impossible.

The study of critical graphs begins with Dirac (1952), and Erdős (1989) records the conjecture. Brown (1992) settled the five-color case. Lattanzio (2002) proved the cases in which $k - 1$ is composite, and Jensen (2002) proved every case with $k \geq 5$. Jensen and Siggers (2012) later showed that a four-color vertex-critical graph can have an arbitrarily small fraction of critical edges. Recent work still identified the complete removal of critical edges at four colors as the remaining case (Skottova and Steiner, 2025; Ferudun, 2026). Theorem 2 supplies that case. Because every vertex of G^* has six neighbors, it also answers the explicit existence question posed by Skottova and Steiner (2025).

4 Policy implications and scope

Due to the possibility of networks identified by Theorem 1, a market designer trying to accurately evaluate a clearing target to address interference should test the removal of entire stations and not only relax individual interference restrictions. While this might not be a concern for small k 's, as small as four, this will already be a concern.

A more indirect implication for market concerns procurement. Again, since removing links one at a time does not reveal the effect of removing a whole station, it is important to not infer that purchasing a station cannot meet the target merely because no individual restriction attached to it reaches the target when relaxed alone. Speculatively, when screening stations for a reverse auction, the market designer should evaluate a station exit as a whole and for each station, the designer should remove that station and all of its interference restrictions, rerun the repacking problem, and ask whether the clearing target can then be met. And then if a station that passes this test it should remain eligible for procurement, even if relaxing each of its interference restrictions separately never meets the target. The auction can then use bids to choose among the station exits that the direct feasibility test identifies. There much design details for such a procurement enterprise but the implication discussed here should be one of them.

5 Conclusion

This paper shows that two ways of addressing interference in a spectrum auction can produce very different results. For every integer $k \geq 4$, there is a network that requires k channels but requires only $k - 1$ after any station leaves the air. Relaxing any single interference restriction, by

contrast, leaves the requirement at k . The economic result follows directly from the proof of Dirac’s conjecture provided in this paper.

It is important to note that this is an existence result, not a claim about every spectrum network. It shows why a market designer cannot generally just infer the value of buying a station from tests that relax its interference restrictions one at a time. Sometimes those tests may be easier to run, but they can miss the joint effect of removing all the restrictions attached to a station and lead to costly design decisions. Procurement decisions should therefore be based on a direct feasibility test: remove the station, remove its associated restrictions, and rerun the repacking problem. The core insight here applies to other networked markets in which removing one participant also removes a bundle of constraints.

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A Exact finite verification

AI-search disclosure. I used OpenAI’s GPT-5.6 Sol model, accessed through ChatGPT, to help design, code, and run the restriction-aware search that found the candidate graph G^* . The search used prior structural restrictions from [Ferudun \(2026\)](#) and then explored larger symmetric graphs. It supplied a candidate, not a proof. Verification starts from the frozen edge list and uses the argument in Section 3.1, two separately written deterministic Python programs, and a formal proof written in Lean 4, a proof assistant. No model output is treated as mathematical evidence. I take responsibility for the construction, code, proofs, and errors.

The construction is distributed with the sorted edge list `candidate60.edgelist`, the adjacency matrix `g60_adjacency.csv`, two exact verifiers, and the paper source. The frozen checksums are

```
candidate60.edgelist 85914250312a4fe8d1a67d747fea2ee1772c9762796af90095efa61fb4c0400c
g60_adjacency.csv fa02371094fbbf032d447b13a0fa16725c7effec90aaa227815dbd4d42761b9d
verify_g60.py 49851c452567c2fdbe865b72dd4bdc2a015d27dcc322790604c70e860fd99e8b
verify_g60_independent.py
                        d13f1cfb928ffd5da7ef9f8efaad1a78e10da95fabd399fc3a630d2ce35615e8
```

With Python 3.10 or later, run

```
python3 verify_g60.py
python3 verify_g60_independent.py
```

from the replication directory. Each successful run ends with **RESULT: PASS**.

The primary verifier first checks the object: 60 labels, 180 unique loopless edges, degree six at each vertex, connectedness, absence of triangles, inverse closure of S , exact agreement between the Cayley definition and the edge list, and preservation of adjacency under all 60 left translations. It then checks the coloring claims using deterministic exhaustive degree-of-saturation (DSATUR) search. The independent verifier reads the same frozen edge list but uses a separate finite-domain constraint solver with singleton propagation and minimum-remaining-value branching. It independently checks the chromatic number, all vertex and edge deletions, the 16 normalized punctured colorings, their neighborhood profile, and the three displayed repair pairs.

Table 1: Finite checks for G^*

Claim	Exact check
Graph structure	60 vertices; 180 distinct edges; no loop; all 60 vertices have degree 6; connected; triangle-free
Adjacency matrix	Frozen CSV hash; binary, symmetric, zero diagonal; exact agreement with the edge list
Cayley specification	Regenerated edge set equals the frozen edge set; every left translation preserves adjacency
Three-colorability of G^*	No proper coloring found after exhaustive search
Normalized colorings of $G^* - 0$	Exactly 16, with the four-star template in Lemma 3
Balanced puncture	Each of colors 0, 1, and 2 occurs twice on $N(0)$ in every normalized coloring
Lean finite certificate	Reported build accepts the balanced-puncture lower bound and order-60 result; 21 exclusion trees replay over 35,967 nodes
Vertex deletions	All 60 are three-colorable and none is two-colorable
Edge deletions	None of all 180 is three-colorable
Three-color defect	Equal to 2; all three displayed two-edge repairs replayed

The direct checks of all vertex and edge deletions are redundant with the balanced-puncture proof and symmetry, but they are useful error detection. Both verifiers are deterministic, use only the Python standard library, and enforce raw edge constraints during search.

A companion Lean 4 source and certificate package is archived at <https://github.com/alexgocardinal/dirac-critical-graph>. The supplied report records successful builds with Lean 4.19.0 and Mathlib 4.19.0. The compact route checks a proper four-coloring, all vertex-deletion witnesses, and the lower bound excluding any three-color assignment with at most one monochromatic edge. The structured route formalizes the balanced-puncture classifier and symmetry transport, culminating in `Dirac60.order60_solution`. Its replay validates 21 exclusion trees—18 unique-color cases and three missing-color cases—over 35,967 nodes and matches the frozen graph and certificate hashes reported in the archive.

The reported axiom audit for the principal order-60 declarations lists only `propext`, `Classical.choice`, `Quot.sound`, and `Lean.ofReduceBool`; it displays no `sorryAx` or problem-specific mathematical axiom. The last dependency records reliance on Lean’s compiler for reflected finite computation rather than on kernel reduction alone. I therefore describe the order-60 result as accepted by Lean in the pinned environment. The two separately implemented Python searches remain additional exact audits, and I make no claim that the search output has been checked by an independent satisfiability-proof certificate. The Lean development covers the new $k = 4$ result. Its universal-decoupling declaration checks the final case split conditional on Jensen’s published $k \geq 5$ existence result; it does not formalize Jensen’s infinite family.

B The normalized punctured colorings

All 16 colorings in Lemma 3 have the template

$$-0**21102120110012022002*1110202021202100*022201111120202021,$$

where positions are labels $0, 1, \dots, 59$, the dash is the deleted vertex, and the four stars vary independently over

$$c(2) \in \{1, 2\}, \quad c(3) \in \{0, 1\}, \quad c(24) \in \{0, 1\}, \quad c(41) \in \{1, 2\}.$$

For ease of hand inspection, Table 2 expands the fixed entries of this template.

Table 2: Fixed color classes in every normalized coloring of $G^* - 0$

Color	Fixed vertex labels
0	1, 7, 11, 14, 15, 18, 21, 22, 28, 30, 32, 36, 39, 40, 42, 46, 53, 55, 57
1	5, 6, 9, 12, 13, 16, 25, 26, 27, 34, 38, 47, 48, 49, 50, 51, 59
2	4, 8, 10, 17, 19, 20, 23, 29, 31, 33, 35, 37, 43, 44, 45, 52, 54, 56, 58
Free	2, 3, 24, 41

The table makes the balanced-puncture calculation visible without trusting the compact string. The six neighbors of 0 are split into the pairs

$$\{5, 26\}, \quad \{30, 55\}, \quad \{33, 37\},$$

with colors 1, 0, and 2, respectively.

C The low-dimensional cases

Under the convention $\chi(\emptyset) = 0$, the one-vertex graph satisfies the conditions at $k = 1$: deleting its only vertex lowers the chromatic number from one to zero, and its edge-deletion condition has no edge on which to operate. This is why the nonvacuous boundary in the paper is stated for networks with an actual interference restriction.

For completeness, this appendix proves the classification used in Corollary 2. A 2-vertex-critical graph containing an edge cannot have a third vertex: deleting that third vertex would leave a 2-chromatic subgraph. It is therefore the two-vertex complete graph, denoted K_2 .

Now let H be 3-vertex-critical. Since H is not two-colorable, it contains an odd cycle C . No vertex lies outside C , because deleting such a vertex would leave the odd cycle. The cycle has no chord: a chord divides it into two cycles of opposite parity, leaving a proper odd cycle that omits some vertex. Hence H is exactly an odd cycle. Deleting any edge of K_2 or of an odd cycle reduces its chromatic number. This proves that complete one-edge immunity is impossible for chromatic numbers 2 and 3.